On Finitary Power Monoids of Linearly Orderable Monoids

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Outline

- Background
- Atomicity and Weaker Notions
- Ascending Chains Condition on Principal Ideals and Weaker Notions
- The Furstenberg Property, IDF, and TIDF

Monoids

Definition: A commutative monoid is a pair $(M, +)$, where M is a set and $+$ is a binary operation satisfying the following conditions:

- $+$ is associative: $a + (b + c) = (a + b) + c$ for all $a, b, c \in M$,
- there exists $e \in M$ such that $e + m = m$ for all $m \in M$,
- $+$ is commutative: $a + b = b + a$ for all $a, b \in M$.

Examples of Monoids:

- Any abelian group (a commutative monoid where every element is invertible).
- $\mathbb{N}_0$, $\mathbb{Q}_{\geq 0}$, and $\mathbb{R}_{\geq 0}$ under addition.
- $\mathbb{N}$, $\mathbb{Q}_{\geq 1}$, and $\mathbb{R}_{\geq 1}$ under multiplication.

Monoids (cont.)

Definition: For a subset S of a monoid M, the monoid generated by S is defined as $\langle S \rangle = \sum_{s \in S} n_s s$ where $n_s \in \mathbb{N}_0$ for all $s \in S$ and is non-zero for finitely many s.

Definition: For a monoid M and $b, c \in M$, we say b divides c (or $b \mid_M c$) if $b + a = c$ for some $a \in M$.

Conventions:

- We will assume all monoids are cancellative commutative monoids: if $a + b = a + c$, then $b = c$ for all $a, b, c \in M$.
- For a monoid $(M, +)$, we will just write M instead.
- M is written additively, unless otherwise specified: the operation is $+$, and the identity is $e = 0$.

Examples:

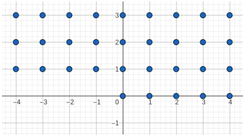
- The monoid generated by $\{1\}$ is $\mathbb{N}_0$.
- The monoid generated by $\{\frac{1}{2^i} : i \in \mathbb{N}\}$ is the set $\{\frac{n}{2^i} : i, n \in \mathbb{N}_0\}$.

Submonoids

Definition: A subset N of a monoid M is called a submonoid of M if $0 \in N$ and N is closed under $+$.

Examples:

- Submonoids of $\mathbb{N}_0$ are called numerical monoids.
  - The monoid $\mathbb{N}_0$ is a numerical monoid.
  - The monoid generated by $\{2, 3\}$ is a numerical monoid.
- Submonoids of $\mathbb{Q}_{\geq 0}$ are called Puiseux monoids.
  - The monoid $\mathbb{Q}_{\geq 0}$ is a Puiseux monoid.
  - The monoid generated by $\{\frac{1}{p} : p \in \mathbb{P}\}$ is a Puiseux monoid.
  - The monoid $\{0\} \cup \mathbb{Q}_{\geq r}$ for $r \in \mathbb{R}^+$ is a Puiseux monoid.
  - The monoid generated by $\{\frac{1}{2}\} \cup \{\frac{1}{3^i} : i \in \mathbb{N}\}$ is a Puiseux monoid.
- The monoid $(\{0\} \times \mathbb{N}_0) \cup (\mathbb{N} \times \mathbb{Z})$ is a submonoid of $\mathbb{Z}^2$.



The Difference Group

Definition: For a monoid M, the difference group (or Grothendieck group), denoted by $\mathrm{gp}(M)$, is the group consisting of equivalence classes $[a-b]$ for $a, b \in M$ where $[a-b] \sim [c-d]$ if $a+d = b+c$ and the binary operation is defined as $[a-b]+[c-d] = [(a+c)-(b+d)]$.

Remark: Note that this is indeed an equivalence relation. Reflexivity and symmetry are trivial, and transitivity follows from cancellativity.

Examples:

- For any numerical monoid $M = \langle S \rangle$, $\mathrm{gp}(M) \cong \gcd(S)\mathbb{Z}$.
- If $M = \mathbb{Z} \setminus \{0\}$ under multiplication, then $\mathrm{gp}(M) \cong \mathbb{Q} \setminus \{0\}$.
- For any group G, $\mathrm{gp}(G) = G$.

Finitary Power Monoids

Definition: The finitary power monoid of a monoid M is the set of non-empty finite subsets of M, denoted by $\mathcal{P}_{\text{fin}}(M)$, with the binary operation of sumset addition. This means that for $S, T \in \mathcal{P}_{\text{fin}}(M)$, we have

$$S + T = \{s + t : s \in S, t \in T\}.$$

Examples:

- In $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$, $\{0,1\} + \{0,1\} = \{0,1,2\}$.
- In $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$, $\{1,3\} + \{0,2,5\} = \{1,3,5,6,8\}$.
- In $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$, $\{0,1\} + \{0,1,2\} = \{0,1\} + \{0,2\} = \{0,1,2,3\}$. Therefore, $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$ is not cancellative.

Totally Ordered Monoids

Definition: A monoid M is torsion-free if $na = nb$ implies $a = b$ for all $n \in \mathbb{N}$ and $a, b \in M$.

Definition: A pair $(M, \leq)$ is a totally ordered monoid if $\leq$ is a total order and $a \leq b$ implies $a + c \leq b + c$ for all $a, b, c \in M$.

Definition: A totally ordered monoid $(M, \leq)$ is positive if $0 \leq m$ for all $m \in M$.

Definition: A positive monoid $(M, \leq)$ is Archimidean if for all non-zero $a, b \in M$, $na > b$ for some $n \in \mathbb{N}$.

Conventions:

- For a totally ordered monoid $(M, \leq)$, we will just write M instead.
- We will assume all monoids are totally ordered.

Examples:

- Any Puiseux monoid under the standard order is a positive Archimidean monoid and therefore totally ordered and torsion-free.
- $\mathbb{Z}/n\mathbb{Z}$ under addition is not torsion-free ($3 \cdot 2 = 0 = 3 \cdot 4$ in $\mathbb{Z}/6\mathbb{Z}$).
- $\mathbb{C}\backslash\{0\}$ under multiplication is not torsion-free.

Totally Ordered Monoids (cont.)

Theorem (Levi, 1913)

An abelian group can be turned into a totally ordered group if and only if it is torsion-free.

Corollary

A commutative monoid can be turned into a totally ordered monoid if and only if it is cancellative and torsion-free.

Corollary

A commutative monoid can be turned into a positive monoid if and only if it is cancellative, torsion-free, and has no invertible elements.

Remark: The corollaries follow from the theorem above through the difference group.

Atoms and Units

Definition: For a monoid M, invertible elements are called units. The group of units is denoted by $\mathcal{U}(M)$. If $a = b + u$ where u is a unit, then a and b are called associates. If $\mathcal{U}(M) = \{0\}$, then M is reduced.

Definition: For a monoid M, some element $a \in M \backslash \mathcal{U}(M)$ is an atom if $a = b + c$ implies b or c is a unit. The set of atoms is denoted by $\mathcal{A}(M)$. If $\mathcal{A}(M) = \emptyset$, then M is called antimatter.

Examples:

- The monoid generated by $\{\frac{1}{2^i} : i \in \mathbb{N}\}$ is reduced and antimatter.
- The atoms of $\mathbb{Z}\backslash\{0\}$ under multiplication are the primes and negative primes, and the units are 1 and -1. Furthermore, x and $-x$ are associates for every $x \in \mathbb{Z}\backslash\{0\}$.

Atomicity

Definition: A monoid M is atomic if every element can be written as a finite sum of atoms and units.

Examples:

- The monoid generated by $\{\frac{1}{2^i} : i \in \mathbb{N}\}$ is not atomic.
- $\mathbb{Z}$ and $\mathbb{N}_0$ under addition and $\mathbb{Z}\setminus\{0\}$ and $\mathbb{N}$ under multiplication are atomic.
- The monoid generated by $\{\frac{1}{p} : p \in \mathbb{P}\}$ is atomic with the set of atoms being $\{\frac{1}{p} : p \in \mathbb{P}\}$.

Maximal Common Divisors (MCD)

Definition: Let M be a monoid and S be a subset of M. An element $m \in M$ is a maximal common divisor (MCD) of S if m is a common divisor of S and no other common divisor d of S exists such that $m \mid_M d$ but $d \nmid_M m$.

Definition: A monoid M is a k-MCD monoid if every subset of M with size k has an MCD. Similarly, M is an MCD monoid if it is k-MCD for all k.

Examples:

- Every finitely generated monoid is MCD.
- The monoid generated by $\{\frac{1}{2^i}, \frac{1}{3} + \frac{1}{2^i} : i \in \mathbb{N}\}$ is not 2-MCD, as $\{1, \frac{4}{3}\}$ does not have an MCD.

A non-MCD Monoid

Example: The monoid $M = \left\langle \{ \frac{1}{2^i}, \frac{1}{3} + \frac{1}{2^i} : i \in \mathbb{N} \} \right\rangle$ is not 2-MCD, as $\{1, \frac{4}{3}\}$ does not have an MCD.

- Every element in $m \in M$ can be written as $\frac{f(m)}{3} + t$ where f is a mapping $f : M \to \mathbb{N}_0$ and $t \in \left\langle \{ \frac{1}{2^i} : i \in \mathbb{N} \} \right\rangle$. (Note that this representation is not unique).
- Notice that $3 \mid_M f(1)$, but any sum of more than 2 elements of the form $\frac{1}{3} + \frac{1}{2^i}$ results in a number greater than 1. So $f(1) = 0$.
- Similarly, $f(\frac{4}{3}) = 1$.
- Any divisor d of 1 satisfies $f(d) = 0$, so if d is a common divisor of $\{1, \frac{4}{3}\}$, then $f(d) = 0$.
- $f(\frac{4}{3} - d) = 1$, so $\frac{1}{3} + \frac{1}{2^i} \mid_M \frac{4}{3} - d$ and therefore $\frac{1}{2^i} \mid_M 1 - d$ for some i.
- So $d_0 = d + \frac{1}{2^{i+1}}$ is also a common divisor of $\{1, \frac{4}{3}\}$, and $d \mid_M d_0$ but $d_0 \nmid_M d$.

MCD and Atomicity

Theorem (D.-Gotti-H.-Li-Schlessinger, PRIMES 2024)

Let M be a monoid, and let $k \in \mathbb{N}$. Then the following are equivalent:

- M is an MCD monoid.
- $\mathcal{P}_{\text{fin}}(M)$ is an MCD monoid.
- $\mathcal{P}_{\text{fin}}(M)$ is a k-MCD monoid.

Theorem (D.-Gotti-H.-Li-Schlessinger, PRIMES 2024)

Let M be an atomic monoid. Then $\mathcal{P}_{\text{fin}}(M)$ is atomic if and only if M is an MCD monoid.

Theorem (Gonzalez-Li-Rabinovitz-Rodriguez-Tirador, 2023), (D.-Gotti-H.-Li-Schlessinger, PRIMES 2024)

There exists an atomic Puiseux monoid M such that $\mathcal{P}_{\text{fin}}(M)$ is not atomic.

Near Atomicity

Definition: Let M be a monoid. Then M is nearly atomic if there exists some $a \in M$ where $a + x$ is atomic for every $x \in M$.

Remark: For a nearly atomic monoid M, the element $a \in M$ mentioned before must be atomic.

Remark: Atomicity $\implies$ near atomicity.

Example:

- Let α be irrational and $\phi : \mathbb{Q}_{\geq 0} \to \mathbb{P}$ be an injective mapping. Then the monoid $M = \langle \{q, \frac{q+\alpha}{\phi(q)} : q \in \mathbb{Q}_{\geq 0}\} \rangle$ under addition is nearly atomic ($\alpha + m$ is atomic for all $m \in M$), but not atomic.

Remark: There exists indeed a nearly atomic Puiseux monoid which is not atomic.

Theorem (D.-Gotti-H.-Li-Schlessinger, PRIMES 2024)

There exists an atomic Puiseux monoid M such that $\mathcal{P}_{\text{fin}}(M)$ is not nearly atomic.

Almost Atomicity and Quasi-Atomicity

Definition: Let M be a monoid.

- M is almost atomic if for every $x \in M$, there exists an atomic $a \in M$ where $a + x$ is atomic.
- M is quasi-atomic if for every $x \in M$, there exists an $a \in M$ where $a + x$ is atomic.

Remark: near atomicity $\implies$ almost atomicity $\implies$ quasi-atomicity.

Examples:

- Let $M_0 = \langle\{\frac{1}{p} : p \in \mathbb{P}\}\rangle$. Then the monoid $M = M_0 \cup \mathrm{gp}(M_0)_{\geq 1}$ is almost atomic but not nearly atomic.
- The monoid $M = \langle\{\frac{1}{2^i} : i \in \mathbb{N}\}\rangle \cup \langle\{\frac{1}{3^i} : i \in \mathbb{N}\}\rangle_{\geq \frac{4}{3}}$ is quasi-atomic but not almost atomic.

Theorem (D.-Gotti-H.-Li-Schlessinger, PRIMES 2024)

There exists an almost atomic monoid M such that $\mathcal{P}_{\mathrm{fin}}(M)$ is not quasi-atomic.

ACCP

Definition: For a subset I of a monoid M, I is an ideal of M if $I + M \subseteq I$. The ideal I is principal if $I = x + M$, denoted by (x), for some $x \in M$.

Definition: A monoid M satisfies the ACCP (ascending chains condition on principal ideals) if every ascending chain of principle ideals $(b_0) \subseteq (b_1) \subseteq (b_2) \subseteq \ldots$ eventually stabilizes (for some $N \in \mathbb{N}$, $(b_m) = (b_n)$ for all $m, n > N$).

Remark: The ACCP is equivalent to every chain of elements $b_0, b_1, \ldots$ where $b_{n+1} \mid_M b_n$ stabilizing up to associate (for some $N \in \mathbb{N}$, b_m and b_n are associate for all $m, n > N$).

Remark: ACCP $\implies$ atomicity.

Examples:

- The monoid generated by $\{\frac{1}{p} : p \in \mathbb{P}\}$ satisfies the ACCP.
- For any rational $q \in (0, 1)$ where $\frac{1}{q} \notin \mathbb{Z}$, the monoid generated by $\{q^n : n \in \mathbb{N}\}$ is atomic but does not satisfy the ACCP.

Almost ACCP and Quasi-ACCP

Definition: Let M be a monoid.

- For an element $s \in M$, we say s satisfies the ACCP if every ascending chain of principal ideals starting from (s) stabilizes.
- M is almost ACCP if for every subset S of M, there exists an atomic common divisor d of S where $s - d$ satisfies the ACCP for some $s \in S$.
- M is quasi-ACCP if for every subset S of M, there exists a common divisor d of S where $s - d$ satisfies the ACCP for some $s \in S$.

Remark: ACCP $\implies$ Almost ACCP $\iff$ quasi-ACCP and atomic. Quasi-ACCP $\implies$ MCD.

Examples:

- The monoid $\mathbb{Q}_{\geq 0}$ is quasi-ACCP but not almost ACCP.
- For any rational $q \in (0, 1)$ where $\frac{1}{q} \notin \mathbb{Z}$, the monoid generated by $\{q^n : n \in \mathbb{N}\}$ is almost ACCP but not ACCP.

Ascent of ACCP and Notions Weaker than the ACCP

Theorem (Gonzalez-Li-Rabinovitz-Rodriguez-Tirador, 2023)

If a monoid M is ACCP, then $\mathcal{P}_{\text{fin}}(M)$ is ACCP.

Theorem (D.-Gotti-H.-Li-Schlessinger, PRIMES 2024)

If a monoid M is almost ACCP, then $\mathcal{P}_{\text{fin}}(M)$ is almost ACCP.

Theorem (D.-Gotti-H.-Li-Schlessinger, PRIMES 2024)

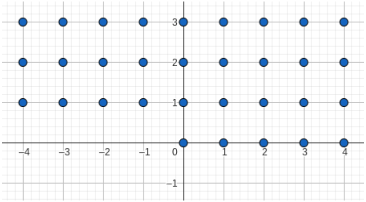
If a monoid M is quasi-ACCP, then $\mathcal{P}_{\text{fin}}(M)$ is quasi-ACCP.

Furstenberg Property

Definition: A monoid M satisfies the Furstenberg property if every non-unit is divisible by at least one atom.

Examples:

- Every atomic monoid satisfies the Furstenberg property.
- Every antimatter monoid which is not a group does not satisfy the Furstenberg property.
- The monoid $(\{0\} \times \mathbb{N}_0) \cup (\mathbb{N} \times \mathbb{Z})$ satisfies the Furstenberg property; however it is not atomic.



Theorem (D.-Gotti-H.-Li-Schlessinger, PRIMES 2024)

If a monoid M satisfies the Furstenberg property, so does $\mathcal{P}_{\text{fin}}(M)$.

Nearly Furstenberg and Almost Furstenberg Properties

Definition: Let M be a monoid.

- M satisfies the nearly Furstenberg property if there exists an element $c \in M$ where for every non-unit $b \in M$, there exists an atom a where $a \mid_M b + c$ but $a \nmid_M c$.
- M satisfies the almost Furstenberg property if for every non-unit $b \in M$, there exists an atomic element c and an atom a where $a \mid_M b + c$ but $a \nmid_M c$.

Remark: Furstenberg $\Longrightarrow$ nearly Furstenberg, almost Furstenberg.

Theorem (Lin-Rabinovitz-Zhang 2023)

There exist infinitely many non-isomorphic Puiseux monoids which satisfy the following properties:

- nearly Furstenberg but not almost Furstenberg;
- almost Furstenberg but not nearly Furstenberg;
- almost Furstenberg and nearly Furstenberg but not Furstenberg.

Quasi-Furstenberg Property

Definition: A monoid M satisfies the quasi-Furstenberg property if for each non-unit $b \in M$, there exists some $c \in M$ and atom $a \in M$ such that $a \mid_M b + c$ but $a \nmid_M c$.

Remark: nearly Furstenberg $\implies$ quasi-Furstenberg and almost Furstenberg $\implies$ quasi-Furstenberg.

Remark: A Puiseux monoid M satisfies the quasi-Furstenberg property $\iff$ M is quasi-atomic $\iff$ M is not antimatter.

Example:

- The monoid generated by $\{\frac{1}{2}\} \cup \{\frac{1}{3^i} : i \in \mathbb{N}\}$ satisfies the quasi-Furstenberg property, however does not satisfy either of the nearly Furstenberg and almost Furstenberg properties.

Quasi-Furstenberg Monoid Example

Example: The monoid $M = \left\langle \{\frac{1}{2}\} \cup \{\frac{1}{3^i} : i \in \mathbb{N}\} \right\rangle$ satisfies the quasi-Furstenberg property, however does not satisfy either of the nearly Furstenberg and almost Furstenberg properties.

- $\frac{1}{2}$ is the only atom in M.
- Any $m \in M$ will be of the form $m = \frac{n_0}{2} + \sum_{i=1}^{k} \frac{n_i}{3^i}$ for $n_i \in \mathbb{N}_0$.
- We can simplify this expression to be $m = \frac{n_0}{2} + \frac{t}{3^i}$ for some $n_0, t, i \in \mathbb{N}_0$.
- If $n_0 > 0$, then $\frac{1}{2} \mid_M m$.
- Otherwise, let $c = N - \frac{t}{3^i}$ where $N = \left\lceil \frac{t}{3^i} \right\rceil$.
- Thus, $\frac{1}{2} \mid_M \frac{t}{3^i} + c$ but $\frac{1}{2} \nmid_M c$.

Ascent of Notions Weaker than the Furstenberg Property

Theorem (D.-Gotti-H.-Li-Schlessinger, PRIMES 2024)

If a monoid M satisfies the nearly Furstenberg property, then $\mathcal{P}_{\text{fin}}(M)$ satisfies the nearly Furstenberg property.

Theorem (D.-Gotti-H.-Li-Schlessinger, PRIMES 2024)

If a monoid M satisfies the almost Furstenberg property, then $\mathcal{P}_{\text{fin}}(M)$ satisfies the almost Furstenberg property.

Theorem (D.-Gotti-H.-Li-Schlessinger, PRIMES 2024)

If a monoid M satisfies the quasi-Furstenberg property, then $\mathcal{P}_{\text{fin}}(M)$ satisfies the quasi-Furstenberg property.

IDF and TIDF

Definition: A monoid M is IDF (irreducible divisor finite) if every element is divisible by finitely many atoms up to associate. The monoid M is TIDF (tight irreducible divisor finite) if it is IDF and satisfies the Furstenberg property.

Examples:

- Every antimatter monoid is IDF.
- Every finitely generated monoid is TIDF.
- The monoid generated by $\{\frac{1}{p} : p \in \mathbb{P}\}$ is not IDF ($\frac{1}{p} \mid_M 1$ for every prime p). However, it is atomic.

Theorem (D.-Gotti-H.-Li-Schlessinger, PRIMES 2024)

If M is a positive Archimidean TIDF monoid, then $\mathcal{P}_{\text{fin}}(M)$ is TIDF.

Theorem (D.-Gotti-H.-Li-Schlessinger, PRIMES 2024)

There exists a positive TIDF monoid M where $\mathcal{P}_{\text{fin}}(M)$ is not IDF.

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End of Presentation

THANK YOU!